CDI-II – Prática 11/5/21

Ficha 9, Ficha 10

Variedades:

1) $F=0$ $\quad F:\mathbb{R}^n \to \mathbb{R}^m,\ m<n,\ C^1$

Normais $\quad$ car $DF(x) = m$

$M = \{x \in \mathbb{R}^n : F(x)=0\}$ Variedade $\dim(M) = n-m$

($n-m$ variáveis independentes)

2) $M = \{(x, y) : y = f(x)\}$

$f:\mathbb{R}^{n-m} \to \mathbb{R}^m,\ C^1$

$M =$ Gráfico de f → Variedade-$(n-m)$

3 – $M = \{ g(x) : x \in \underset{\substack{\uparrow \\ \text{aberto}}}{T} \subset \mathbb{R}^{n-m} \}$

tangentes

$g : T \to \mathbb{R}^n$, C^1, injectiva,

car $Dg(t) = n-m = \dim(M)$.

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1-a) $y = x^3 = f(x)$

$f : \mathbb{R} \to \mathbb{R}$, C^1

Parametrização

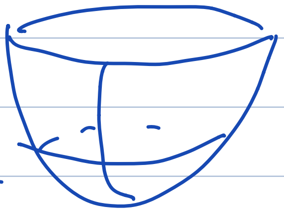
$g(x) = (x, x^3)$

$g : \mathbb{R} \to \mathbb{R}^2, C^1$

Gráfico de f → Variedade-1

imagem de g

1-d) $\underbrace{z = x^2 + y^2}$ < 1



$z = f(x,y)$, $f : \mathbb{R}^2 \to \mathbb{R}, C^1$

Parametrização,

$$g(x,y) = (x, y, x^2+y^2)$$

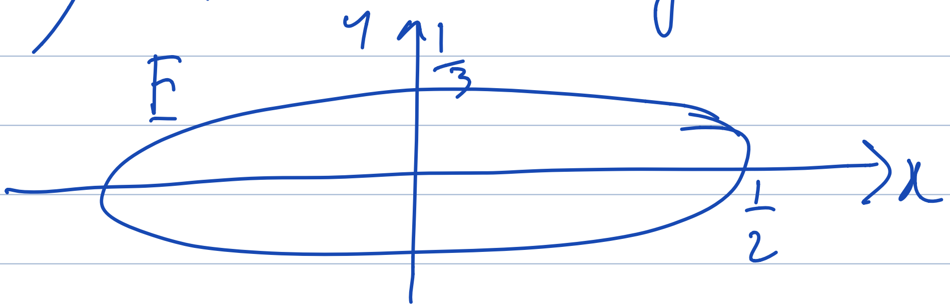
$$T = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$$ aberto

$$g : T \longrightarrow \mathbb{R}^3,\ C^1,$$ injectiva

$$\text{Car}\, Dg(x,y) = 2 \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2x & 2y \end{bmatrix}$$

——— II ———

1-b) $E : 4x^2+9y^2=1$



$$E:\ F(x,y) = 4x^2+9y^2-1 = 0$$

$$F: \mathbb{R}^2 \to \mathbb{R},\ C^1$$

$$DF(x,y) = \begin{bmatrix} 8x & 18y \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\forall (x,y) \in E$$

$$\Longrightarrow \text{car}\, DF(x,y) = 1 \Rightarrow \dim(E) = 1$$

Parametrizar: $2x = \cos t$

$3y = \operatorname{sen} t$

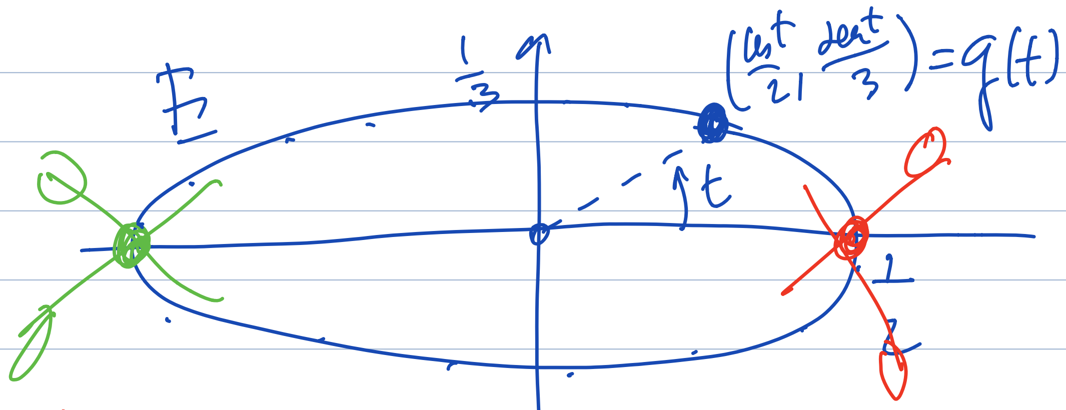
$0 < t < 2\pi$

aberto

$$g(t) = (x(t), y(t)) = \left(\frac{\cos t}{2}, \frac{\operatorname{sen} t}{3}\right)$$

$$g:]0, 2\pi[\to \mathbb{R}^2,\ C^1,\ \text{injectiva}$$

$$\text{car}\, Dg(t) = \text{car}\left(\begin{bmatrix} -\frac{\operatorname{sen} t}{2} \\ \frac{\cos t}{3} \end{bmatrix}\right) = 1$$



$$E \setminus \left\{\left(\frac{1}{2}, 0\right)\right\} = g\Big(\underbrace{\overset{\downarrow}{]0}, \overset{\downarrow}{2\pi}[}_{T}\Big)$$

$$\boxed{h(t) = \left(\frac{\cos t}{2}, \frac{\operatorname{sen} t}{2}\right) \qquad -\pi < t < \pi}$$

$$E \setminus \left\{\left(-\frac{1}{2}, 0\right)\right\} = h\big(]-\pi, \pi[\big)$$

$$\underset{\text{objeto}}{t} \xrightarrow{g,\,h} \Big(\underset{x(t)}{\frac{\cos t}{2}}, \underset{y(t)}{\frac{\operatorname{sen} t}{2}}\Big)$$

1-c) $M: F(x,y,z) = y^2+z^2-1 = 0$

$F: \mathbb{R}^3 \longrightarrow \mathbb{R}$, C^1

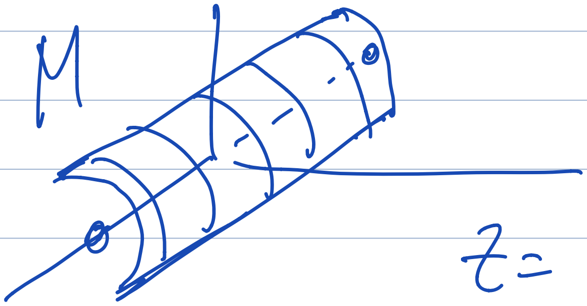
$\boxed{y>0,\ z>0,\ |x|<1}$

$$DF(x,y,z) = \begin{bmatrix} 0 & 2y & 2z \end{bmatrix}$$

$$\neq \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \quad 0 \neq 1$$

$$\forall (x,y,z) \in M$$

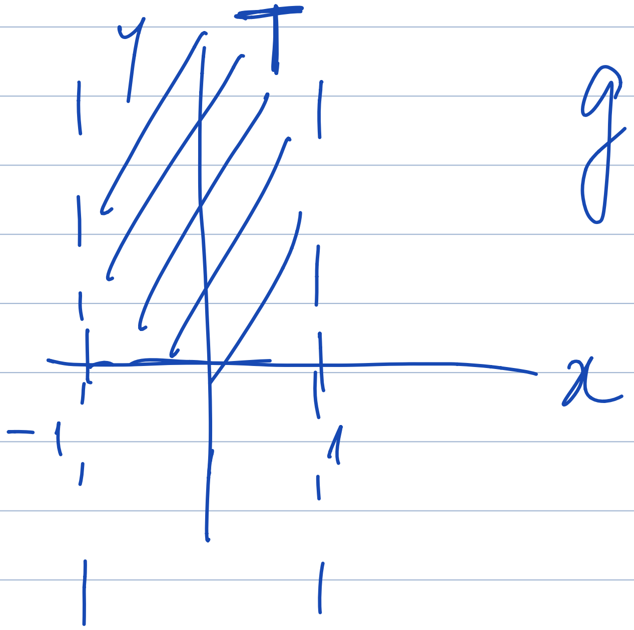
$$\Rightarrow \dim(M) = 3-1 = 2 //$$



$z = \sqrt{1-y^2}$

ou $y = \sqrt{1-z^2}$

$$g(\widehat{x,y}) = (\widehat{x,y}, \sqrt{1-y^2})$$



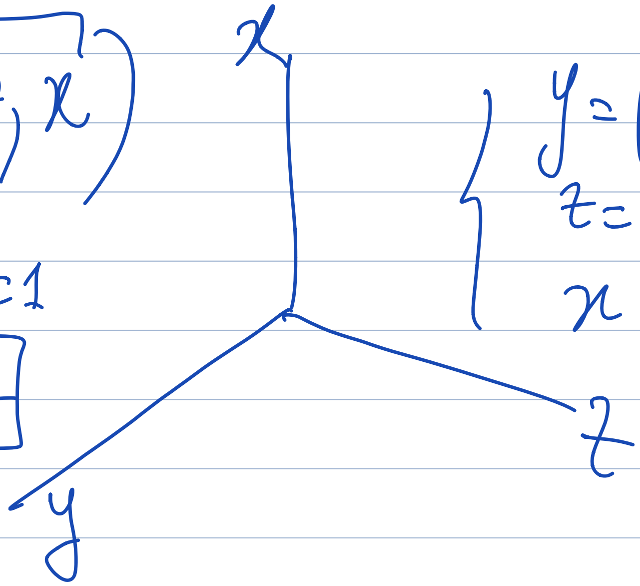
$g: T \longrightarrow \mathbb{R}^3, C^1$

injective

$$Dg(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ \dots & \dots \end{bmatrix}$$

ou

$(\rho, \overline{\theta, x})$

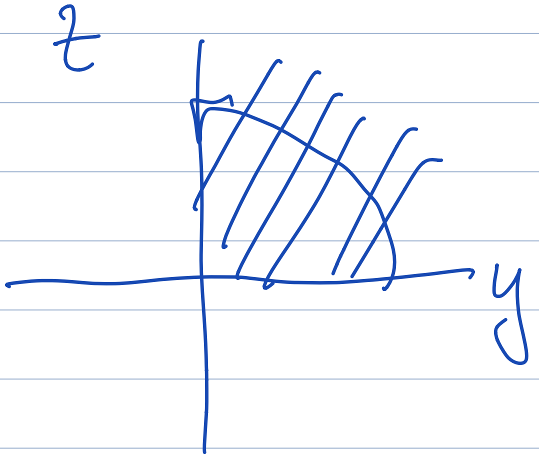


$$\begin{cases} y = \rho\cos\theta \\ z = \rho\,\mathrm{sen}\,\theta \\ x = x \end{cases}$$

$y^2 + z^2 = 1$

$\boxed{\rho = 1}$

$$h(\theta, x) = (x, \cos\theta, \operatorname{sen}\theta)$$



$$\theta = \arctan \frac{z}{y}$$

$$\boxed{\begin{matrix} 0 < \theta < \frac{\pi}{2} \\ -1 < x < 1 \end{matrix}} \; T$$

$h : T \to \mathbb{R}^3$, C^1 injetiva

$$Dh(\theta, x) = \begin{bmatrix} 0 & 1 \\ -\operatorname{sen}\theta & 0 \\ \cos\theta & 0 \end{bmatrix}$$

$$\underset{\text{lin. ind.}}{\uparrow \quad \uparrow}$$

1-b) $M:$ $\boxed{\begin{array}{c} z = x^2+y^2 \\ x+y=1 \end{array}}$ $\begin{array}{l} x>0 \\ y>0 \\ z<1 \end{array}$

$$\left\{\begin{array}{l} t = x^2+y^2 \\ \\ y = 1-x \end{array}\right. \quad \Bigg| \quad \begin{array}{l} z = x^2+(1-x)^2 \\ \\ y = 1-x \end{array}$$

$$\left(x, \underbrace{1-x, \; x^2+(1-x)^2}_{}\right) = g(x)$$

$x>0$

$$\left\{\begin{array}{l} F(x,y,z) = \left(z-x^2-y^2, \; x+y-1\right) = (0,0) \\ \\ DF(x,y,z) = \begin{bmatrix} -2x & -2y & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}\right.$$

$\det = -1 \neq 0$

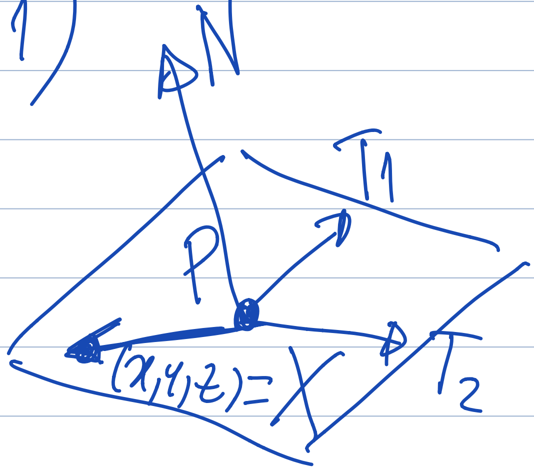
$\underline{\underline{\dim(M) = 1}}$

5 – $g(x,y) = (x, y, xy)$

└→ 2 tangentes

$$Dg(x,y)_{(1,1)} = \begin{array}{c} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ y & x \end{bmatrix} \\ \uparrow \;\; \uparrow \\ \text{tangentes} \end{array} \underset{(x=1,\,y=1)}{=} \begin{array}{cc} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \\ \uparrow \quad \uparrow \\ T_1 \quad T_2 \end{array}$$

$P = (1,1,1)$



$$(X-P)\cdot N = 0$$

↓

eq. do plano tangente

$$N = (a,b,c)$$

$$\begin{cases} N \cdot T_1 = 0 \\ N \cdot T_2 = 0 \end{cases} \qquad \begin{cases} a + c = 0 \\ b + c = 0 \end{cases} \quad \begin{array}{|l} c = -a \\ c = -b \end{array}$$

$$N = (a,b,c) = (a, a, -a) = a(1,1,-1)$$

eq. do plano tangente em $(1,1,1) = P$

$$\boxed{(X - P) \cdot N = 0}$$

$$(x-1, y-1, z-1) \cdot N = 0$$

$$(x-1) + (y-1) - (z-1) = 0$$

$$\boxed{x + y - z = 1}$$

Ficha 10 – Extremos condicionados

$f : \mathbb{R}^n \to \mathbb{R}, \; C^1$

$F(x) = 0 \;, \quad F : \mathbb{R}^n \to \mathbb{R}^m, \; m < n, \; C^1$

$\text{car}\, DF(x) = m$

extremos de f sujeitos às condições $F = 0$:

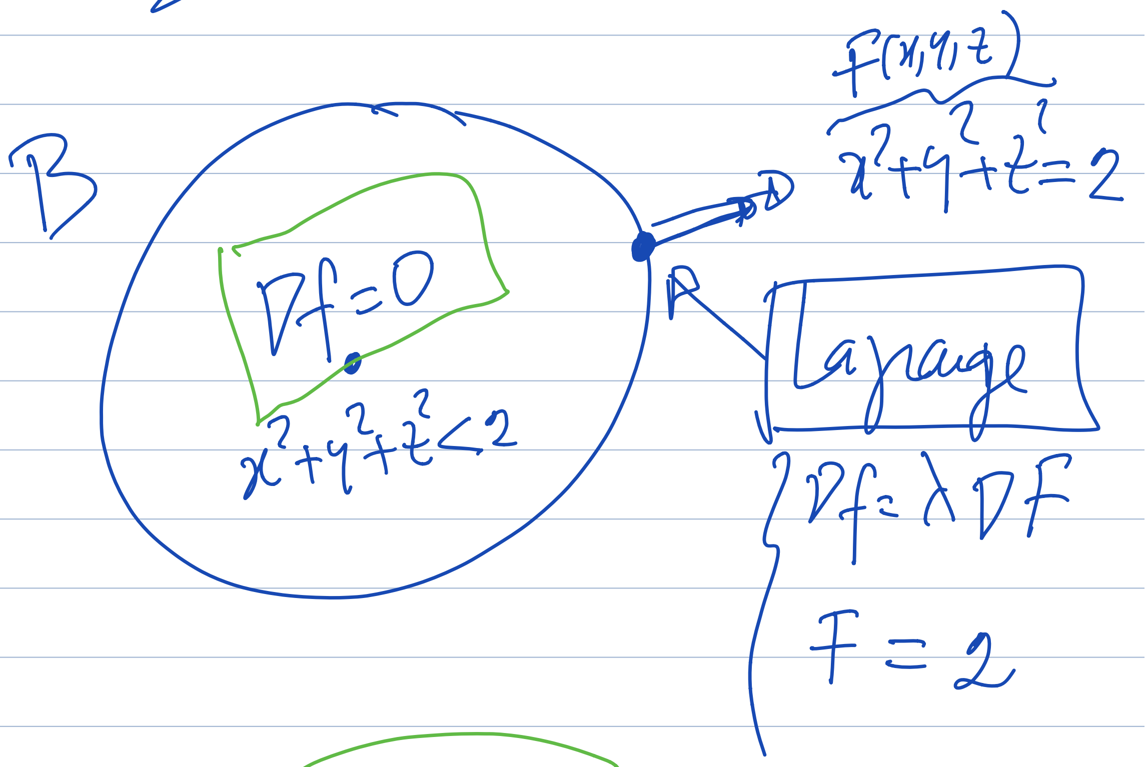
Solução (Lagrange)

$$\begin{cases} \nabla f = \lambda_1 \nabla F_1 + \cdots + \lambda_m \nabla F_m \\ F_1 = 0 \\ \vdots \\ F_m = 0 \end{cases}$$

✓

2- $f(x,y,z) = z^2 - x - y$

$B: \; x^2+y^2+z^2 \le 2$



$$\nabla f(x,y,z) = \overbrace{(-1,-1,2z)} \neq (0,0,0)$$

No interior de B não há extremos de f.

Extremos de f estão na fronteira.

$$\begin{cases} (-1,-1,2z) = \lambda(2x,2y,2z) \\ x^2+y^2+z^2=2 \end{cases}$$

✓

$$\begin{cases} -1 = 2\lambda x \\ -1 = 2\lambda y \\ 2z = 2\lambda z \\ x^2+y^2+z^2=2 \end{cases} \qquad \begin{cases} \boxed{0 = 2\lambda(x-y)} \\ \underline{\qquad\qquad\qquad} \\ \boxed{2z(1-\lambda)=0} \\ \underline{\qquad\qquad\qquad} \end{cases}$$

✓ ✓

$$\begin{cases} \boxed{\lambda=0} \vee y=x \\ \underline{\qquad\qquad} \\ \lambda=1 \vee z=0 \\ x^2+y^2+z^2=2 \end{cases}$$

etc...

↑

4 — $\underline{(\text{distância})^2 \text{ à origem}}$:

$f(x,y,z) = x^2+y^2+z^2$ máx. de f na linha ?

$g(t) = (\overset{\frown}{\cos t}, \overset{\frown}{\operatorname{sen} t}, \boxed{\operatorname{sen}(2t)})$, $t \in \mathbb{R}$

$= (x(t), y(t), z(t))$

↳ <u>2 equações</u> :

$\operatorname{sen}(2t) = 2 \operatorname{sen} t \cos t$

$z = 2xy$

$$\begin{cases} E_1 \rightarrow \boxed{x^2+y^2=1} \\ E_2 \rightarrow \boxed{z = 2xy} \end{cases}$$

... Lagrange

$$\begin{cases} (2x, 2y, 2z) = \lambda_1(2x, 2y, 0) + \lambda_2(-2y, -2x, 1) \\ x^2 + y^2 = 1 \\ z = 2xy \end{cases}$$

$$\begin{cases} 2x = 2\lambda_1 x - 2\lambda_2 y \\ 2y = 2\lambda_1 y - 2\lambda_2 x \\ 2z = \lambda_2 \\ x^2 + y^2 = 1 \\ z = 2xy \end{cases}$$

$$\begin{cases} 2(x+y) = 2\lambda_1(x+y) - 2\lambda_2(x+y) \\ \_\_\_\_ \\ \_\_\_\_ \\ \_\_\_\_ \\ \_\_\_\_ \end{cases}$$

etc ...